國立陽明交通大學
113學年度碩士班考試入學招生試題

科目：線性代數與離散數學(8102)

系所班別：資訊聯招

考試日期：113年2月2日 第 2 節

【不可使用計算機】

＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

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First you need to write down your answer clearly and then explain how to compute the answers. You also need to answer the questions in order. Do not jump around.

1. **(5 points)** Solve the following puzzle.

一名醫護人員說：『醫院裡的醫護人員，包括我在內，總共是16名醫生與護士。並且下面四條狀況都成立。並且如果把我排除不計的話，下面四條狀況仍然都成立。』

一、護士多於醫生。
二、男醫生多於男護士。
三、男護士多於女護士。
四、至少有一位女醫生。

請問說話的人是甚麼性別和職務?

請注意，每個人要麼是男性，要麼是女性，不可以又是男又是女。每個人要麼是醫生、要麼是護士，不可以又是醫生又是護士。

2.

a. **(5 points)** Consider the following non-decreasing sequence of natural numbers $a_1, a_2, a_3, \cdots$: 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5,⋯. Note that there are exactly k^2 occurrence of k. Define a function $f(n)$ as the largest integer m such that $a_m = n$. For example, $f(3) = 14$ and $f(4) = 30$. What is $f(f(n))$?

b. **(5 points)** Find all non-negative integers a and b so that $\sqrt{a} + \sqrt{b} = \sqrt{2009}$.

3.

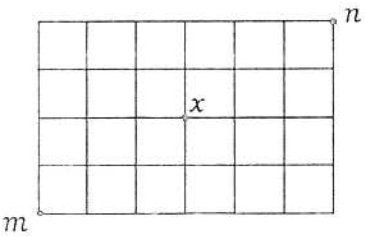
a. **(5 points)** The *Goldbach's conjecture* says, "every even integer greater than 2 is a sum of two prime numbers." Please translate Goldbach's conjecture into a logical formula using $\forall$, $\exists$, *and* ($\wedge$), *or* ($\vee$), *not* ($\neg$), *implies* ($\rightarrow$) and *if-and-only-if* ($\leftrightarrow$), and the four basic integer arithmetic operators (+,−, *,/). Note that / is integer division, for example, $5/2 = 2$.

b. **(5 points)** A *bit string* is a string of 0 and 1 digits. For instance, 0011101110 is a bit string of length 10. How many bit strings of length 10 contain either five consecutive 1's or five consecutive 0's?

4. **(11 points)** We want to count the number of paths from vertices m to n in the following graph. Each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward (No moves to the left or downward are allowed.)

a. **(3 points)** What is the number of paths from vertices m to n?

b. **(4 points)** What is the number of paths from vertices m to n that do not go through vertex x?

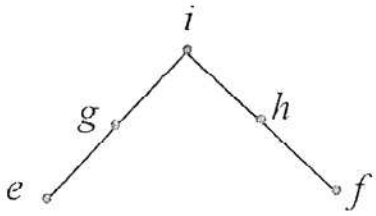


c. **(4 points)** Let P be the set of all vertices in the above graph. Define a binary relation R on P such that $(p, q) \in R$ if we need either one unit to the right or one unit upward to move from vertices p to q. Is (P, R) a partially ordered set? Why or why not?

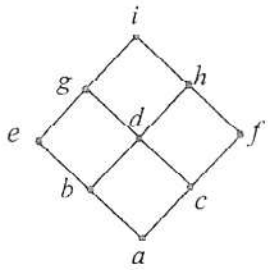
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5. **(6 points)** If we run topological sorting on the following Hasse diagrams, how many different results can we get?

(2 points) (a)



(4 points) (b)



6. **(8 points)** Suppose that we already have the following theorem.
If G is a connected planner simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.
Using this theorem to prove that if G is a connected planner simple graph with at least three vertices, then G has a vertex of degree not exceeding five.

7. **(12 points)** Let

$$\boldsymbol{A} = \begin{bmatrix} 0 & -3 & 3 & 9 \\ 2 & 3 & -1 & 1 \\ 1 & 3 & -2 & 2 \\ 4 & 10 & -5 & 1 \end{bmatrix}$$

a. **(6 points)** Find the $\boldsymbol{A^{-1}} = ?$
b. **(6 points)** Find the $\boldsymbol{PA=LU}$ factorization for A, where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is a lower triangular matrix with unit diagonal, and $\boldsymbol{U}$ is an upper triangular matrix.

8. **(8 points)** Given 4 points (t, b): (0, 20), (1, 0), (2, 0), and (3, 0), what is the sum of squared error $\|e\|^2$ if we fit the closest quadratic polynomial $b = C + Dt + Et^2$ by the least square approximation?

9. **(5 points)** Prove that if a square matrix $\boldsymbol{A}$ is invertible, then $(\boldsymbol{A^{-1}})^T = (\boldsymbol{A^T})^{-1}$. (Please leave it blank if you don't know the correct answer, or you will get at most minus 5 points (until questions 7, 8, and 9 are 0 points) for the wrong answer. 不會寫請留白，答錯最多倒扣 5 分，扣至 7, 8, 9 題 0 分為止。)

10. Given

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2\mathbf{A} & 2\mathbf{A} \\ \mathbf{A} & 3\mathbf{A} \end{bmatrix}, \text{ please find}$$

a. **(5 points)** the eigenvalues of **A**,
b. **(5 points)** the corresponding eigenvectors of A, and
c. **(5 points)** the eigenvalues of **B**.

11. **(5 points)** Let $\mathbf{C} = \begin{bmatrix} 9 & 3 & -6 \\ 3 & 4 & 1 \\ -6 & 1 & 7 \end{bmatrix} = \mathbf{L}\mathbf{L}^T$, where L is a 3x3 lower triangular matrix. Please find the **L**.

12. **(5 points)** Let $\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix} = \mathbf{F}^H\mathbf{F}$, where F is a 3x3 Hermitian matrix. Please find the **F**.