國立臺灣師範大學 115 學年度碩士班招生考試試題

科目：數學基礎　　適用系所：資訊工程學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Some notations:

- A *vector* refers to a column vector with real entries, for example, $\boldsymbol{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} \in \mathcal{R}^3$.
- A linear transformation where the domain and codomain equal $\mathcal{R}^n$ is called a **linear operator** on $\mathcal{R}^n$.
- The **orthogonal complement** of a nonempty subset S of $\mathcal{R}^n$, denoted by $S^{\perp}$, is the set of all vectors in $\mathcal{R}^n$ that are orthogonal to every vector in S.

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1. (a) (8 points) Suppose that the reduced row echelon form of matrix A is $R = \begin{bmatrix}1 & -1 & 0 & -2 & 0 & 2\\0 & 0 & 1 & 3 & 0 & -4\\0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}$. Determine A if the second, fourth and sixth columns of A are $\boldsymbol{a_2} = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$, $\boldsymbol{a_4} = \begin{bmatrix}1\\6\\-2\end{bmatrix}$ and $\boldsymbol{a_6} = \begin{bmatrix}-1\\-7\\3\end{bmatrix}$, respectively.

   (b) (8 points) T is a linear operator on $\mathcal{R}^3$ defined by $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}cx_1\\2x_1 - 3x_2 + 2x_3\\-3x_1 - x_3\end{bmatrix}$ for some scalar c. Determine all the values of c for which T is not diagonalizable.

2. (11 points) Suppose that A is an $m \times n$ matrix with linearly independent columns. Any factorization $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal set in $\Re^m$ and R is an $n \times n$ upper triangular matrix, is called a QR factorization of A. Determine the matrices Q and R in a QR factorization of A, where $A = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$.

3. (11 points) Given a solution set, a subspace, $W = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \Re^3;\ x_1 + x_2 - x_3 = 0 \text{ and } x_1 - x_2 + 3x_3 = 0\right\}$ and a vector $\boldsymbol{u} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$, find the unique vectors w in W and z in $W^{\perp}$ such that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{z}$, where $W^{\perp}$ is the orthogonal complement of W.

4. (12 points) Find an orthogonal operator T on $\mathcal{R}^3$ such that $T(\boldsymbol{v}) = \boldsymbol{w}$ where $\boldsymbol{v} = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\\0\end{bmatrix}$ and $\boldsymbol{w} = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\-2\\1\end{bmatrix}$.

Notice: In the following questions, all graphs are assumed to be simple and undirected. For a graph G, we use $V(G)$ to denote its vertex set.

5. (5 points) Write down the definition: Two graphs G and H are isomorphic if _____.

6. The *completement* of a graph G is a graph H with $V(G) = V(H)$ such that for $u, v \in V(G)$, u and v are adjacent in G if and only if u and v are not adjacent in H. A graph G is self-complementary if G and its complement are isomorphic.

   i. (10 points) Prove or disprove: Every self-complementary graph is connected.

   ii. (20 points) Please list (by drawing) **all** 5-vertex graphs that are self-complementary. You have to argue the correctness. (Hint: It would be helpful to list all valid degree sequences.)

7. (15 points) Find a particular solution of the following system of congruences:

$$x \equiv 59 \pmod{314}$$
$$x \equiv 97 \pmod{159}$$
$$x \equiv 32 \pmod{263}$$

It is known that $x = 6549784$ is a solution of the following system:

$$x \equiv 58 \pmod{314}$$
$$x \equiv 97 \pmod{159}$$
$$x \equiv 32 \pmod{263}$$

You can use arithmetic expressions (in which additions, subtractions, multiplications, and integers are allowed) to present your answer.