國立臺灣師範大學 111 學年度碩士班招生考試試題

科目：數學基礎　　適用系所：資訊工程學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. (10 分) Find $f(n)$ when $n=3^k$, where f satisfies the recurrence relation $f(n)=5f(n/3)+8$ with $f(1)=2$.

2. (10 分)
   (1) How many different strings can be made from the letters in *WANNANNASSA*, using all the letters?
   (2) What is the coefficient of $x^{103}y^{97}$ in the expansion of $(2x-3y)^{200}$?

3. (10 分)
   (1) Let S and R be relations on a set A represented by the matrices

$$\mathbf{M}_S=\begin{bmatrix}0&1&1\\1&1&1\\1&0&0\end{bmatrix} \quad \text{and} \quad \mathbf{M}_R=\begin{bmatrix}0&1&0\\1&1&0\\1&1&1\end{bmatrix}.$$

   What is the matrix representing $R\circ S$?
   (2) Given the function

$$f\colon \mathbf{Z}\times\mathbf{Z}\to\mathbf{Z},\quad f(m,n)=m+n,$$

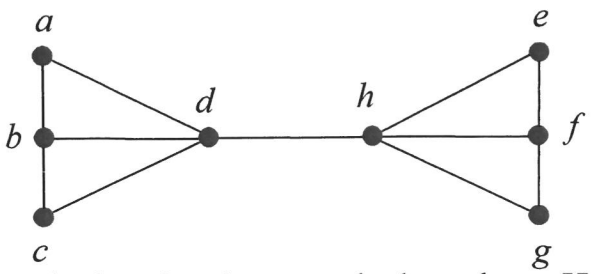
   write down its domain, codomain, domain of definition, and the set of values for which it is undefined. Also, determine whether it is a total function.

4. (10 分)
   (1) Suppose that the domain consists of all propositions. Express the following statement using logical operators, predicates, and quantifiers.

   The conjunction of two tautologies is a tautology.

   (2) Find 7^{123} **mod** 11.

5. (10 分)



   (1) Determine whether the given graph above has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.
   (2) Determine whether the given graph above has a Hamilton path. If so, find such a path. If it does not, give an argument to show why no such path exists.

Related notations:

- We work with column vectors, for example, $\boldsymbol{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} \in \Re^3$.
- Let A be an $m\times n$ matrix. The function T_A: $\Re^n \to \Re^m$ defined by $T_A(\boldsymbol{x})=A\boldsymbol{x}$ for all $\boldsymbol{x}$ in $\Re^n$ is called the **matrix transformation induced** by A.
- Let T be a linear operator on $\Re^n$ and $\mathcal{B} =\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be a basis for $\Re^n$. The matrix $[[T(\boldsymbol{b}_1)]_{\mathcal{B}} \quad [T(\boldsymbol{b}_2)]_{\mathcal{B}} \quad \cdots \quad [T(\boldsymbol{b}_n)]_{\mathcal{B}}]$ is called the **matrix representation of T with respect to $\mathcal{B}$** or the **$\mathcal{B}$-matrix of T**, denoted by $[T]_{\mathcal{B}}$.

6. (8 分) Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent vectors in $\Re^3$.
   (1) (4 分) Find the reduced row echelon form of $A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3 \quad \boldsymbol{a}_4]$ given that $\boldsymbol{a}_1 = \boldsymbol{u},\ \boldsymbol{a}_2 = \boldsymbol{u} + 2\boldsymbol{v},\ \boldsymbol{a}_3 = 2\boldsymbol{u} + 2\boldsymbol{v},\ \boldsymbol{a}_4 = \boldsymbol{v}.$
   (2) (4 分) Find the rank and nullity of A.

7. (4+4 分) Let A be a 3×3 matrix such that $\det A = -7$. Evaluate the determinant of each of the following matrices, where a_{ij} represents the (i, j) entry of A:
   (a) $\begin{bmatrix} a_{11}+2a_{31} & a_{12}+2a_{32} & a_{13}+2a_{33} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 2a_{31}-5a_{21} & 2a_{32}-5a_{22} & 2a_{33}-5a_{23} \end{bmatrix}$; (b) $\begin{bmatrix} -a_{13} & -2a_{11} & a_{12} \\ -a_{23} & -2a_{21} & a_{22} \\ -a_{33} & -2a_{31} & a_{32} \end{bmatrix}$

8. (10 分) Let T be a linear operator on $\Re^3$ defined by $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}3x_1 + x_3\\x_1 + x_2\\-x_1 - x_2 + 3x_3\end{bmatrix}$. Calculate $[T]_{\mathcal{B}}$, the matrix representation of T with respect to the basis $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ where $\boldsymbol{b}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, and $\boldsymbol{b}_3 = \begin{bmatrix}2\\1\\1\end{bmatrix}$.

9. (12 分) Let T be a linear operator T on $\Re^3$ defined by $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_2 + 2x_3\\2x_1 + 2x_3\\2x_1 + 2x_2\end{bmatrix}$. Find an *orthonormal* basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix. Also show $[T]_{\mathcal{B}}$.

10. (12 分) Given a vector $\boldsymbol{v} = \begin{bmatrix}1\\5\\1\\-1\end{bmatrix}$ and a subspace W which is the solution set of
    $x_1 + x_2 - x_3 + x_4 = 0.$
    $x_1 - x_2 + 3x_3 + x_4 = 0.$
    (1) (8 分) Find the orthogonal projection matrix P_W.
    (2) (4 分) Find the vector $\boldsymbol{w}$ in W that is closest to $\boldsymbol{v}$.