國立臺灣師範大學 110 學年度碩士班招生考試試題

科目：數學基礎　　適用系所：資訊工程學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分

Related notations:

- We work with column vectors, for example, $\boldsymbol{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} \in \mathcal{R}^3$.
- A **linear transformation** where the domain and codomain equal $\mathcal{R}^n$ is called a **linear operator** on $\mathcal{R}^n$.
- The **transpose** of a matrix A is denoted by A^T.
- An $n \times n$ matrix is an **orthogonal matrix** (or **orthogonal**) if its columns form an *orthonormal* basis for $\mathcal{R}^n$. A linear operator on $\mathcal{R}^n$ is called an **orthogonal operator** (or **orthogonal**) if its standard matrix is an orthogonal matrix.

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1. Given matrices $A = \begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 4 & 3 & 6 & 0\\ -7 & -3 & -2 & 1\end{bmatrix}$ and $B = \begin{bmatrix}-1 & 3 & -2 & 9\\ 2 & -2 & -3 & 8\\ -3 & 4 & 1 & -1\\ 1 & -1 & 2 & -1\end{bmatrix}$,
   (a) (5 points) compute the determinant of the matrix $A^{-1}B^T$ and
   (b) (3 points) compute the determinant of the matrix $A^{-1}C$ where $C = 3B^T$.

2. (6 points) For the matrix $A = \begin{bmatrix}0 & -1 & 1\\ 1 & 2 & -3\\ 3 & 1 & -2\\ -1 & 0 & 4\end{bmatrix}$, determine the dimension of each subspace: (a) Col A, (b) Row A, and (c) Null A^T.

3. (12 points) Find an explicit formula for the reflection operator T_w of $\mathcal{R}^3$ about the plane W with equation $x - 2y + 3z = 0$.

4. (12 points) An inconsistent system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is given where $A = \begin{bmatrix}1 & 2 & -1\\ 1 & -1 & 2\\ 2 & 1 & 1\end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix}1\\3\\1\end{bmatrix}$. Use the method of least squares to obtain the vector $\boldsymbol{z}$ for which $\|A\boldsymbol{z} - \boldsymbol{b}\|$ is a minimum.

5. (12 points) Find an orthogonal operator T on $\mathcal{R}^3$ such that $T(\boldsymbol{v_1}) = \boldsymbol{w_1}$ and $T(\boldsymbol{v_2}) = \boldsymbol{w_2}$, where $\boldsymbol{v_1} = \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$, $\boldsymbol{v_2} = \frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix}$, $\boldsymbol{w_1} = \frac{1}{7}\begin{bmatrix}2\\3\\6\end{bmatrix}$ and $\boldsymbol{w_2} = \frac{1}{7}\begin{bmatrix}6\\2\\-3\end{bmatrix}$.

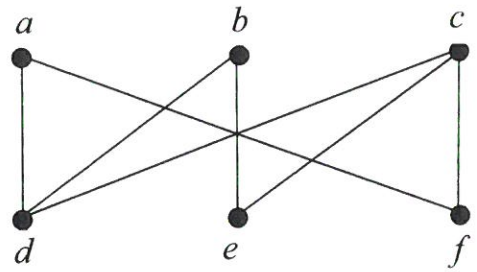
6. (10 points) Let the set A=$\{x \mid x \in \mathbf{Z}$ and $x > 20\}$. Determine whether the set A is finite, countably infinite, or uncountable. If it is countably infinite, exhibit a one-to-one correspondence between the set of positive integers and the set A. If it is finite, show its cardinality.

7. (10 points) What is the solution of the recurrence relation
$a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$
with $a_0 = 6$ and $a_1 = 8$?

8. (10 points)
(1) Determine whether the relation represented by the following zero–one matrix is an equivalence relation.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(2) Determine whether the given graph is planar. If so, draw it so that no edges cross.



9. (10 points)
(1) There are 36 different time periods during which classes at a university can be scheduled. If there are 769 different classes, how many different rooms will be needed?
(2) Encrypt the message RUNNING REINDEERS using blocks of five letters and the transposition cipher based on the permutation of {1, 2, 3, 4, 5} with σ(1)=3, σ(2)=5, σ(3)=1, σ(4)=2, σ(5)=4. For this problem, use the letter X as many times as necessary to fill out the final block of fewer than five letters.

10.(10 points)
(1) Find a counterexample, if possible, to the universally quantified statement, $\forall x \exists y(y^2 = x)$, where the domain for all variables consists of all integers.
(2) Let $\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$. Find $(\mathbf{A}^{-1})^3$.